Computational Cost of Direct Numerical Simulations

To perform a direct numerical simulation (DNS) of some fluid flow, the number of grid points must accurately capture all scales of motion. This means that the number of grid points N in one direction must span the largest scale of the motion at the integral length scale L, but must also capture the smallest scales of motion at the Kolmogorov length scale η. This means at minimum,

$$N \approx \frac{L}{\eta} \,. \tag{1}$$

The Kolmogorov length scale (in m) is defined as

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} , \tag{2}$$

where ν is the viscosity in $\mathrm{m^2\,s^{-1}}$, and ε is the turbulence dissipation rate in $\mathrm{m^2\,s^{-3}}$. This rate ε can be written as

$$\varepsilon = \frac{u^3}{L} \,, \tag{3}$$

where u is velocity in $\mathrm{m\,s^{-1}}$. Substituting these definitions into Equation 1, we see that

$$\begin{aligned}
N &\approx \frac{L}{\eta} && (4)\\
&\approx L\left(\frac{\varepsilon}{\nu^3}\right)^{1/4} && (5)\\
&\approx \frac{L}{\nu^{3/4}}\left(\frac{u^3}{L}\right)^{1/4} && (6)\\
&\approx \frac{L^{3/4}u^{3/4}}{\nu^{3/4}} && (7)\\
&\approx \mathrm{Re}^{3/4} \,. && (8)
\end{aligned}$$

We then estimate that the number of floating-point operations in one direction (N) is proportional to the number of grid points (three dimensions) plus the number of time steps (one "dimension"). Thus, the computational cost of a DNS with N^4 grid points scales with Re^3.